Malthus: The curse of fixed factors

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Economic growth: Theory and Empirical Methods

Introduction

- You have seen, that for **more than a thousand years**, we observed almost no economic growth.
- One of the earliest economists thinking about income growth was the English economist Thomas Malthus who focused on the absence of growth during the middle ages.
- Malthus (1798) observed in the UK that whenever land became more productive, the population would increase and food production per person would remain constant in the long run.
- One particular example is the introduction of the potato in Ireland after 1750. A potato field produces two to three times more nutrition than a weed field (Ireland becomes more productive). After some time, the population of Ireland tripled, and living standards remained unaltered.

Modeling production

The main output in the middle ages was food. We will assume that we can aggregate all types of food into a single output good Y.

The main factors of production were labor, land, and the technology used on the land.

- Labor was relatively homogeneous, and we assume we can aggregate it into a single measure L that may change over time.
- The amount of land, X, is fixed.
- We will consider the cases where technology, $B(t)$, is a constant and when it changes over time.

Production II

Consider the following Cobb Douglas production function:

$$Y(t) = B(t)X^{\alpha}L(t)^{1-\alpha}, \tag{1}$$

where $\alpha < 1$ is the relative importance of land in the production process. To make the notation more compact, we can write this as

$$Y(t) = A(t)L(t)^{1-\alpha}, \tag{2}$$

with $A(t) = B(t)X^{\alpha}$ being the efficient land. Hence, when land becomes three times more productive, $A(t)$ scales up by a factor of three.

Production III

Important in the production process is that we have diminishing marginal returns to labor. The marginal returns are

$$\frac{\partial Y(t)}{\partial L(t)} = (1-\alpha)A(t)L(t)^{-\alpha} > 0. \quad (3)$$

These marginal returns become smaller as we increase labor, i.e., the second derivative is negative:

$$\frac{\partial^2 Y(t)}{\partial^2 L(t)} = -\alpha(1-\alpha)A(t)L(t)^{-\alpha-1} < 0. \quad (4)$$

As a result, output per worker, $y(t) = \frac{Y(t)}{L(t)} = A(t)L(t)^{-\alpha}$, is decreasing in labor:

$$\frac{\partial y(t)}{\partial L(t)} = -\alpha A(t)L(t)^{-\alpha-1} < 0. \quad (5)$$

Production and income

To understand household decisions, we need to know their income. Given the agrarian economy, we will assume that each farm belongs to a household. Hence, total household income equals total household production.

Population growth

At the heart of Malthus theory is that the only thing that leads people to have fewer (surviving) children than the natural birth rate, Z, is too low income. Low income leads to famines, diseases, and wars thus reducing the population growth rate. Accordingly, we model the population growth rate as increasing in income per person:

$$n(t) = \frac{\dot{L}(t)}{L(t)} = Z - \frac{1}{y(t)}. \tag{6}$$

Note, as $y(t) \to \infty$, population growth approaches Z.

Steady state

We start with the case were technology is fixed and $A(t) = A$. Assume a steady state exists where output per capita is constant, i.e., its growth rate is zero. To obtain the growth rate of output per capita, we start with log output per capita, and take the derivative with respect to time and use the fact that the derivative of a variable in logs with respect to time is the growth rate of that variable:

$$y(t) = AL(t)^{-\alpha} \tag{7}$$

$$\ln y(t) = \ln A - \alpha \ln L(t) \tag{8}$$

$$\frac{\dot{y}(t)}{y(t)} = -\alpha \frac{\dot{L}(t)}{L(t)} = 0 \tag{9}$$

Steady state II

Now substitute in the law of motion for labor:

$$\frac{\dot{y}(t)}{y(t)} = -\alpha \frac{\dot{L}(t)}{L(t)} = 0 \tag{10}$$

$$-\alpha \left(Z - \frac{1}{y^*} \right) = 0 \tag{11}$$

$$y^* = \frac{1}{Z}. \tag{12}$$

The steady state indeed exists. The endogenous variable y depends only on exogenous parameters (constants).

Steady state III

$$y^* = \frac{1}{Z}. \tag{13}$$

- Countries with a lower natural birth rate will be richer as fewer people work on the amount of available efficient land.
- Output per capita does not depend on the fixed factor A!

Steady state IV

We can also solve for the amount of labor in the steady state:

$$y^* = \frac{1}{Z} \tag{14}$$

$$A\left(L^*\right)^{-\alpha} = \frac{1}{Z} \tag{15}$$

$$L^* = (AZ)^{\frac{1}{\alpha}} \tag{16}$$

The steady state population increases in the natural birth rate and the amount of efficient land. Countries with more land or a better technology to work that land will have larger populations in the long run.

Key insights

- A better technology will in the long run increase the population.
- The increase in the population will make each worker less productive because the amount of efficient land cannot be endogenously altered.
- As a result, a better technology will not increase output per worker in the long run.

General solution and convergence

The steady state is just one possible level of output per worker. A general solution tells us the level of output per worker, $y(t)$, for an initial starting point, $y(0)$, and the time that has passed, t. To analyze these dynamics, we go back to the dynamics of output per capita over time:

$$\frac{\dot{y}(t)}{y(t)} = -\alpha \frac{\dot{L}(t)}{L(t)} \tag{17}$$

Output per capita growth is negatively proportional to the growth rate in labor. As more workers arrive, the fixed factor land loses productivity.

General solution and convergence II

Now substitute the law of motion for labor to obtain a first-order differential equation in $y(t)$:

$$\frac{\dot{y}(t)}{y(t)} = -\alpha\left[Z - \frac{1}{y(t)}\right] \tag{18}$$

$$\dot{y}(t) = -\alpha Z y(t) + \alpha. \tag{19}$$

This is similar to what we have seen before but for the constant α. To deal with it, define

$$u(t) = \dot{y}(t) = -\alpha Z y(t) + \alpha \tag{20}$$

$$\dot{u}(t) = -\alpha Z \dot{y}(t) \tag{21}$$

$$\Rightarrow \dot{u}(t) = -\alpha Z u(t). \tag{22}$$

General solution and convergence III

As we have seen, the solution is given by

$$u(t) = u(0)\exp(-\alpha Z t) \tag{23}$$

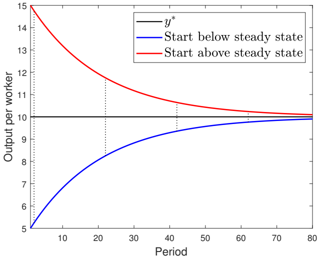
Now substitute back the definition of $u(t)$:

$$-\alpha Z y(t) + \alpha = [-\alpha Z y(0) + \alpha]\exp(-\alpha Z t) \tag{24}$$

$$y(t) = \underbrace{\frac{1}{Z}}_{y^*} + \left[y(0) - \frac{1}{Z}\right]\exp(-\alpha Z t). \tag{25}$$

- When the economy starts in steady state, $y(0) = y^*$, we have that $y(t) = y^*$.
- When the economy starts above its steady state, $y(0) > y^*$, we have that $y(t) > y^*$.
- However, as $t \mapsto \infty$ $y(t) \mapsto y^*$.

The shape of convergence



$$y(t) - \underbrace{\frac{1}{Z}}_{y^*} = \left[y(0) - \frac{1}{Z}\right] \exp(-\alpha Z t). \tag{26}$$

- $y(t) - y^*$ is an exponential growth process: converges at rate $-\alpha Z$ to zero.
- In words: the absolute gap between $y(t)$ and its steady state vanishes at a constant rate αZ.

The shape of convergence II

To obtain the growth rate of $y(t)$, consider again the differential equation:

$$\dot{y}(t) = -\alpha Z y(t) + \alpha \tag{27}$$

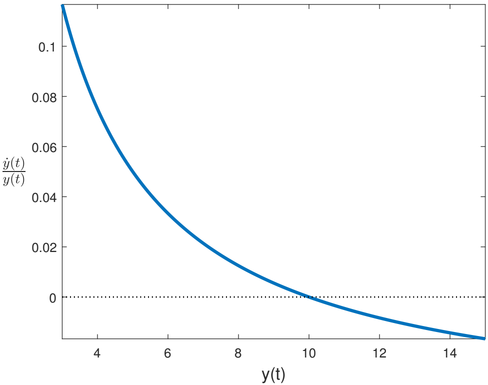
$$\frac{\dot{y}(t)}{y(t)} = -\alpha Z + \frac{\alpha}{y(t)} \tag{28}$$

Note, the growth rate is 0 if $y(t) = \frac{1}{Z} = y^*$. It is a decreasing, convex function in $y(t)$, and

$$y(t) \mapsto 0 \quad \frac{\dot{y}(t)}{y(t)} \mapsto \infty$$

$$y(t) \mapsto \infty \quad \frac{\dot{y}(t)}{y(t)} \mapsto -\alpha Z.$$

The shape of convergence III



The (absolute) growth rate is higher the further the economy is away from steady state.

Dynamics of labor over time

To obtain the dynamics of labor, substitute $y(t) = AL(t)^{-\alpha}$:

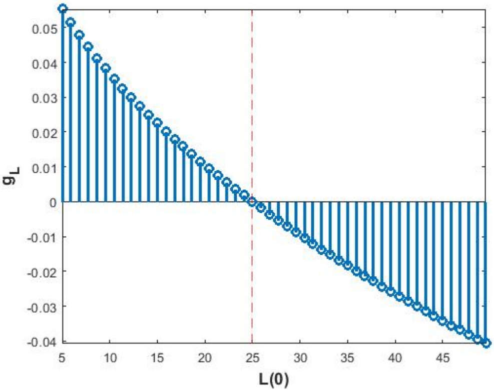
$$y(t) = \frac{1}{Z} - \left[\frac{1}{Z} - y(0)\right] \exp(-\alpha Z t) \tag{29}$$

$$AL(t)^{-\alpha} = \frac{1}{Z} - \left[\frac{1}{Z} - AL(0)^{-\alpha}\right] \exp(-\alpha Z t) \tag{30}$$

$$\frac{1}{L(t)^{\alpha}} = \frac{1}{AZ} + \left[\frac{1}{L(0)^{\alpha}} - \frac{1}{AZ}\right] \exp(-\alpha Z t) \tag{31}$$

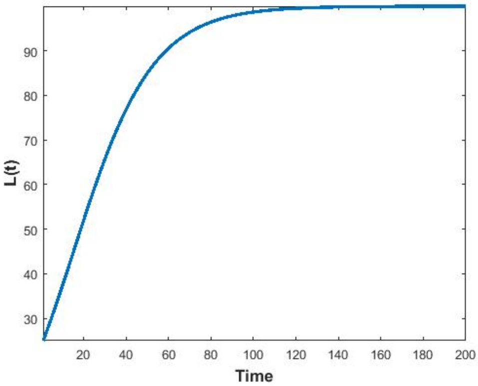
- Over time, $L(t)^{\alpha}$ converges to its steady state AZ.
- The gap between $\frac{1}{L(t)^{\alpha}}$ and its steady state vanishes at a constant rate αZ.
- The (absolute) growth rate is higher the further the economy is away from steady state.

Dynamics of labor over time II



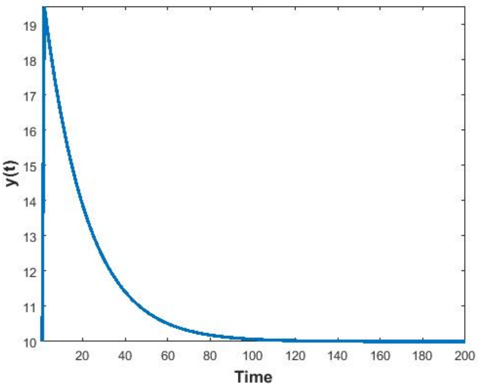
- Population growth is the largest the further we are below steady state.
- The reason is that the further we are below steady state, the higher is income per person.

A one time increase in productivity



- As output increases, the economy can sustain a larger population.
- As seen before, convergence to the new steady state takes place in a concave fashion.

A one time increase in productivity II



- Initially, doubling productivity doubles output per capita.
- As the population increases, output per capita reverts back to its (unchanged) steady state.

Continuous productivity growth

So far, we have only considered a one time change in the level of productivity. Instead, assume now a constant exponential growth rate:

$$A(t) = A(0)\exp(gt) \Rightarrow \frac{\dot{A}}{A} = g. \tag{32}$$

Hence, the growth rate of output per worker depends now on the growth rate of technology and the growth rate of labor:

$$y(t) = A(t)L(t)^{-\alpha} \tag{33}$$

$$\ln y(t) = \ln A(t) - \alpha \ln L(t) \tag{34}$$

$$\frac{\dot{y}(t)}{y(t)} = \frac{\dot{A}(t)}{A(t)} - \alpha \frac{\dot{L}(t)}{L(t)} \tag{35}$$

Continuous productivity growth II

$$\frac{\dot{y}(t)}{y(t)} = g - \alpha \frac{\dot{L}(t)}{L(t)} \tag{36}$$

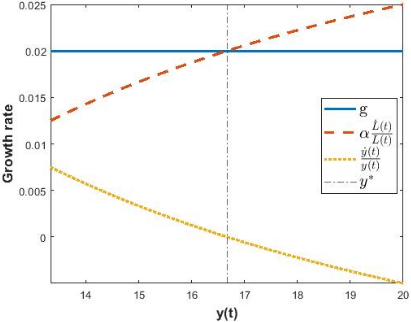
Output per worker will grow if $g > \alpha \frac{\dot{L}(t)}{L(t)}$. This will occur at low levels of output per worker. Vice versa, output per worker will fall when $g < \alpha \frac{\dot{L}(t)}{L(t)}$ which occurs at high levels of output per worker. Finally, we have a steady state in output per worker when

$$g = \alpha \frac{\dot{L}(t)}{L(t)} = \alpha \left(Z - \frac{1}{y^*} \right) \tag{37}$$

$$y^* = \frac{\alpha}{\alpha Z - g}. \tag{38}$$

Note, the steady state only exists when $g < \alpha Z$.

Continuous productivity growth III



Note, continuous productivity growth does not lead to continuous output per capita growth. All it does is raise the steady state level of output per capita.

General solution and convergence

We can solve again the differential equation to obtain a solution for any $y(t)$:

$$\frac{\dot{y}(t)}{y(t)} = g - \alpha \frac{\dot{L}(t)}{L(t)} \tag{39}$$

$$\frac{\dot{y}(t)}{y(t)} = g - \alpha \left[Z - \frac{1}{y(t)} \right] \tag{40}$$

$$\dot{y}(t) = (-\alpha Z + g) y(t) + \alpha. \tag{41}$$

Define

$$u(t) = \dot{y}(t) = -(\alpha Z - g) y(t) + \alpha \tag{42}$$

$$\Rightarrow \dot{u}(t) = -(\alpha Z - g) u(t) \tag{43}$$

$$u(t) = u(0) \exp(-(\alpha Z - g) t) \tag{44}$$

General solution and convergence II

Substituting for $u(t)$:

$$-(\alpha Z - g)y(t) + \alpha = [-(\alpha Z - g)y(0) + \alpha] \exp(-(\alpha Z - g)t) \quad (45)$$

$$y(t) = \underbrace{\frac{\alpha}{\alpha Z - g}}_{y^*} + \left[y(0) - \frac{\alpha}{\alpha Z - g}\right] \exp(-(\alpha Z - g)t). \quad (46)$$

- Output per worker converges over time to its steady state level $\frac{\alpha}{\alpha Z - g}$.
- The absolute gap between $y(t)$ and its steady state vanishes at a constant rate $\alpha Z - g$.
- Hence, technological progress not only changes the steady state, but also slows down convergence to the steady state.

Policy implications

"Yet in all societies, even those that are most vicious, the tendency to a virtuous attachment (i.e., marriage) is so strong, that there is a constant effort towards an increase of population. This constant effort as constantly tends to subject the lower classes of the society to distress and to prevent any great permanent amelioration of their condition.", Malthus (1798)

- Malthus concluded that birth control, postponement of marriage, and celibacy for poor people would be possible solutions.

- The United Nations Sustainable Development Goals include family planning.

- In the Malthus world, monetary transfers to the poor have only transitory benefits on their economic well-being.

What is different today

You have seen that we have constant economic growth since 200 years. To break the logic of the Malthus model, we need to break at least one of its two key assumptions:

1. Population grows with income per person as higher income allows us to approach the natural birth rate:
   - Methods of contraception allow us today to choose any birth rate we desire. In all developed economies, it lies well below the natural birth rate.
2. The factors of production other than labor cannot be altered endogenously:
   - Today, fertilizers and new constructions make land less finite. Moreover, much of our production today requires other forms of capital that can be altered when the population increases.

References

MALTHUS, T. R. (1798): *An essay on the principle of population*, J. Johnson.